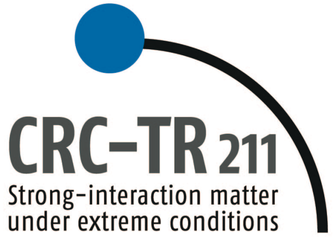
Discrete, Non-Abelian Gauge Theory on a Quantum Annealer

Michael Fromm, NHR-Computational Physics Symposium,
Nov 7-8th, 2022



QCD phase diagram

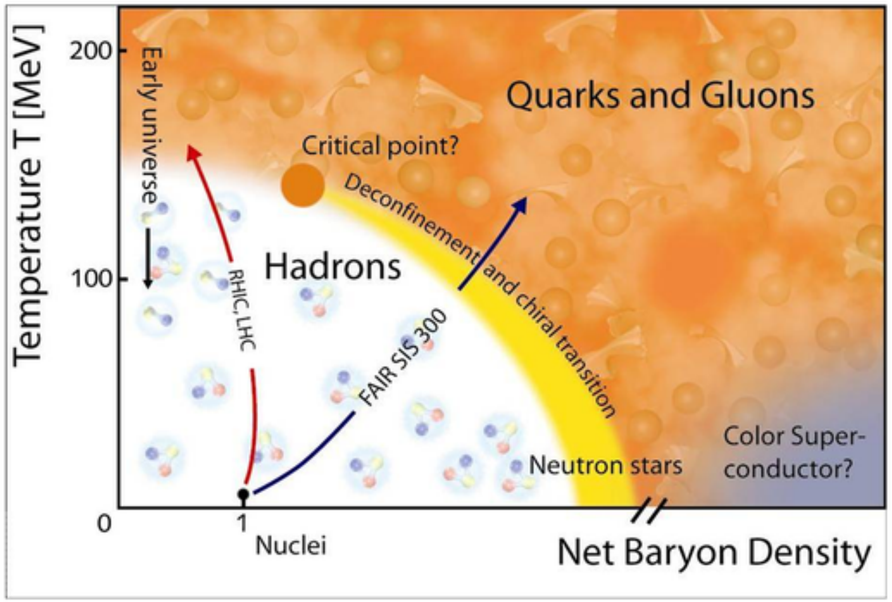


Fig. © GSI

- Large part of the phase diagram accessible only by non-perturbative methods, i.e. via simulation
- Quantum nature of system: Simulation methods relying on importance sampling fail (“Sign Problem”)
  - Restricted parameter space of phase diagram directly numerically accessible (essentially only T-axis)
  - Alternative methods: Tensor networks, quantum simulation (digital, analog)

Quantum Systems as Simulation Platforms

- **Analog quantum computation**
  - Encode the properties (Hamiltonian H) of your quantum system of interest in another “fully-controlled” quantum system,
  - Once prepared, time evolution happens continuously (hence analog and not step-wise / digital)
  - Often “special purpose”, i.e. limited to specific interactions but scalable
  - Ex.: Ultracold atoms on optical lattices, trapped ions
- **Digital QC**
  - Ex: Superconducting (transmon) qubits on a “chip”, e.g. latest IBM architecture (IBM “Eagle”, Nov.’21) with 127 qubits
  - Computations on set of qubits defined as sequential (hence digital) application of gates on this set
  - Programmable (in this sense universal) but currently hampered by noise (**N**oisy**I**ntermediate**S**cale**Q**uantum era)
  - Simulation of quantum system via e.g.
    - State preparation and measurement (in the context of variational approaches)
    - Quantum dynamics by stroboscopic time evolution (“Trotterization”)
- **Other**: Quantum Annealer, D-Wave Advantage with > 5K qb

[1] Parra-Rodriguez, A. et al., Phys. Rev. A 101, 022305 (2020)
[2] Wiese, Uwe-Jens, Phil.Trans.A.Math.Phys.Eng.Sci. 380 (2021)

Quantum Simulation for HEP

Digital Quantum Simulation

- Use **Hamiltonian Formulation of Lattice QCD**

$$\hat{H}_{KS} = \hat{H}_{FB} + \hat{H}_M + \hat{H}_{B,mag} + \hat{H}_{B,kin}$$

- Fermions $\hat{H}_{FB}, \hat{H}_M$ … contrary to Monte-Carlo approaches not problematic, efficient formulations for quantum simulations known (Jordan-Wigner in $d = 1$, Cirac-Verstraete and others[1] for $d > 1$)

- Gauge bosons $\hat{H}_{FB}, \hat{H}_{B,mag}, \hat{H}_{B,kin}$ have to be approximated as their Hilbert Space is infinite, additional constraint Gauss' law, efficient approaches under development[2,3]

- Count of logical qubits ? Rough estimate[4] $\mathcal{O}(10^5)$

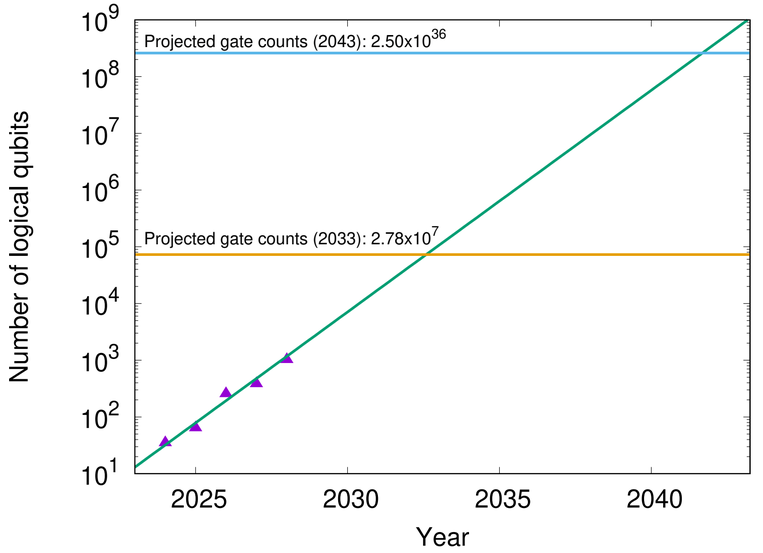


Fig. taken from [4]

[1] Irmejs, R. et al., arXiv:2206.08909 [quant-ph]
[2] Davoudi, Z. et al., Phys. Rev. D 104, 074505 (2021)
[3] Gustafson, E. et al., Snowmass 2021 LOI TF10-07
[4] Kan, A. et al., arXiv:2107.12769 [quant-ph]

Quantum Simulation for HEP

Thermal States

To access finite temperature range, need to prepare mixed state

$$\rho \sim \exp(-\beta \hat{H}) \sim \sum_i e^{-\beta E_i} |i\rangle\langle i|,$$

But current quantum devices are built to prepare pure states $|\psi\rangle$ … approaches ?

- Thermal Pure Quantum States[1] $|\beta, N\rangle = e^{-\frac{\beta}{2}\hat{H}} |\psi_R\rangle$, with random Haar-state $|\psi_R\rangle$ prepared from random circuits
  - Involves non-unitary transformation (approx. by e.g. QITE[2] or embedding into larger system)
- Variational approaches "β-VQE" or "VQT"[3]
  - Classical machine for sampling and optimisation $\mathscr{L} \rightarrow min$
  - Quantum device
    - stores/prepares state
    - computes $\langle H \rangle$ efficiently
  - **But:** Entropy S_θ does not need to be prepared classically

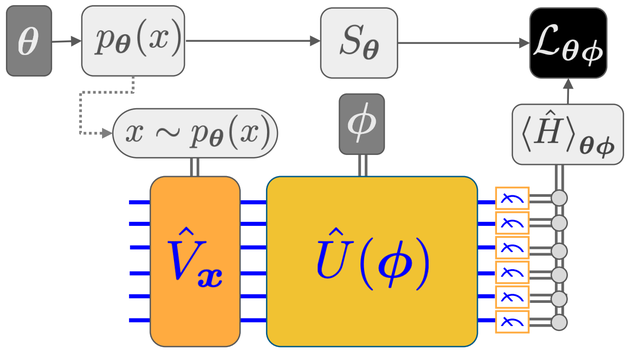


Fig. taken from [3]

[1] Powers et al., arXiv: 2109.01619 [quant-ph]
[2] Motta et al., Nature Physics 16, 205-210 (2020)
[3] Verdon et al., arXiv: 1910.02071 [quant-ph]

Quantum Simulation for HEP

Thermal States

- “Sampling” by mid-circuit measurement “qVQT”[1]

$$F(\theta, \phi) = \langle \hat{H} \rangle - T \langle \hat{S} \rangle \rightarrow min$$

- Can it be generalised to gauge theories ?
  - Additional constraint of Gauss’ law

$$G(x) \, | \psi \rangle = | \psi \rangle,$$

  - Circuits have to preserve gauge invariance
  - State after measurement to be invariant state
  - Parameter count θ_i, ϕ_i has to remain modest
  - …

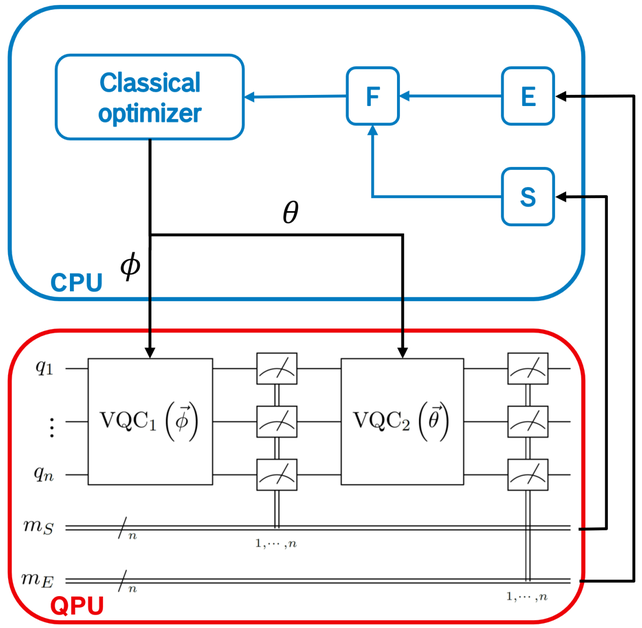


Fig. taken from [1]

[1] Selisko, J. et al., arXiv: 2208.07621 [quant-ph]

Quantum Simulation for HEP

Quantum Annealing

QA[1] using Quantum Adiabatic Principle

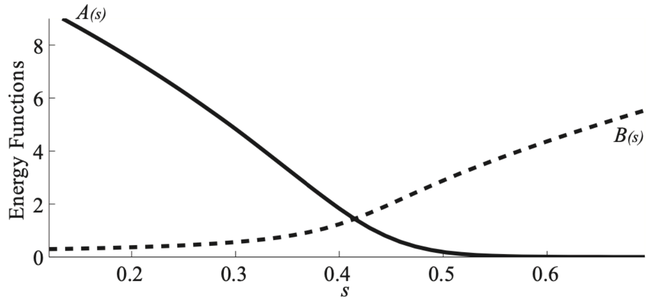
$$H_{QA} = A(s)\, H_I + B(s)\, H_{KS}$$

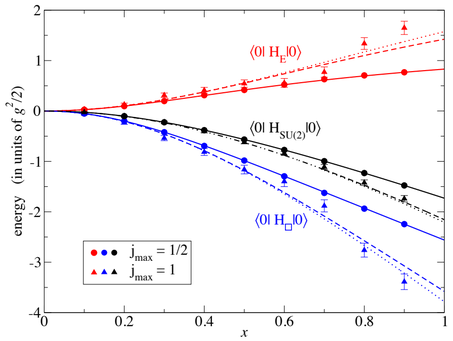
At end of annealing schedule ground-state of target Hamiltonian H_{KS} retrieved .

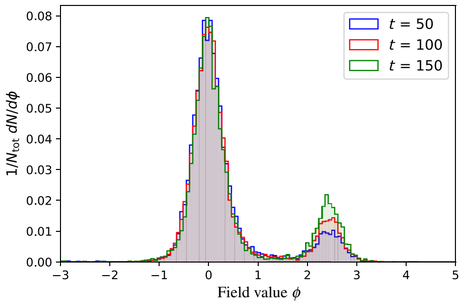
- Ground-state calculation for (truncated) SU(N), N=2,3 gauge theory[2,3] via variational principle

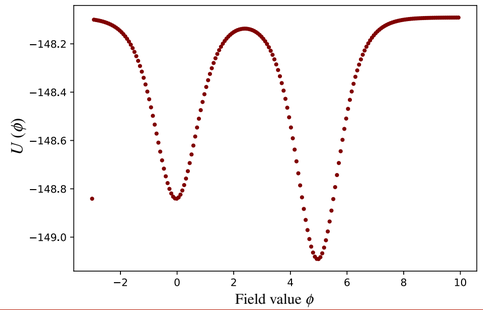
$$E_0 \leq \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle},$$

- Annealer as quantum lab for e.g. tunnelling processes of quantum field theories[4]









[1] Catherine C. McGeoch, Synthesis Lectures on Quantum Computing 2014 5:2, 1-93
[2] Rahman, S. et al., Phys. Rev. D 104, 034501 (2021)
[3] Illa, Marc and Savage, Martin J., arXiv: 2202.12340 [quant-ph]
[4] Abel, S. and Spannowsky, M., PRX Quantum 2, 010349 (2021)

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Gauge Theory Digitization
Quantum Annealing

Recapitulation: Basic Ingredients for Digitization of Gauge Theory (for general Lie group G)

Kogut-Susskind[1] Hamiltonian H_{KS} on Hilbert Space $\mathscr{H} = \bigotimes_{\ell} \mathscr{H}_{\ell}$,

$$H_{KS} = \lambda_E \sum_{x,\mu,\alpha} (E^{\alpha}(x,\mu))^2 - \lambda_B \sum_{p} \mathrm{Tr}(U_p + U_p^{\dagger}) \; , \; \hat{U}^j_{mn} = \int \mathrm{dg}\, D^j_{mn}(g)\, |\, g\rangle\langle g\, |$$

with quadratic Casimir components $(E^{\alpha})^2$ and link operators $\hat{U}^j_{mn}$ building the plaquette p where $g \in G$.

Gauge Invariance

$\theta^R_g\, |\, h\rangle = |\, hg^{-1}\rangle$ and $\theta^L_g\, |\, h\rangle = |\, g^{-1}h\rangle$ have $\theta^L_g\, \hat{U}^j_{mn}\, \theta^{L,\dagger}_g = D^j_{mk}(g) U^j_{kn}$ and $\theta^R_g\, \hat{U}^j_{mn}\, \theta^{R,\dagger}_g = U^j_{mk} D^j_{kn}(g)$

Now define[2] gauge transformation via $\hat{\Theta}_g(x) = \prod_{i,o} \hat{\theta}^{L,o}_g(x)\, \hat{\theta}^{R,\dagger,i}_g(x)$ and **Gauss's law** $\hat{\Theta}_g(x)\, |\, \psi\rangle = |\, \psi\rangle$.

[1] Kogut, John and Susskind, Leonard, Phys. Rev. D 11 (1975)
[2] see e.g. Bender, J. et al., New J. Phys. 20, 093001 (2018)

Gauge Theory Digitization

Quantum Annealing

One can approximate[1] the infinite dimensional $\mathscr{H}_\ell$ …

- … in the **group element basis**, i.e. given the continuous gauge group G with states "labeled" by $\{|g\rangle\}_{g\in G}$, one could choose a suitable finite subset[2] / discrete subgroup G'[3], i.e. make $\mathscr{H}_\ell$ finite dimensional.

- … in the **representation basis,** i.e. change of basis $\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{|G|}} D^j_{mn}(g)$ with representation states $\{|jmn\rangle\}$ (if G is a compact Lie group), where $j = 0, \frac{1}{2}, 1 \ldots$ labels the irreducible representation and m, n label multiplicity within that irrep. One then truncates at a suitable j_{max}.

Other approaches[4,5], involving a change in *dof…*

- **Quantum Link Models**
- **Loop-String-Hadron Formulation**

[1] Zohar, Erez, Phil. Trans. R. Soc. A.380 20210069 (2022)
[2] Hartung, T. et al., arXiv: 2201.09625 [hep-lat], see also talk by Timo Jakobs
[3] Alam, M. et al., Phys.Rev.D.105 (2022)
[4] Gustafson, E. et al., Snowmass 2021 LOI TF10-07
[5] Davoudi, Z. et al., Phys. Rev. D 104, 074505 (2021)

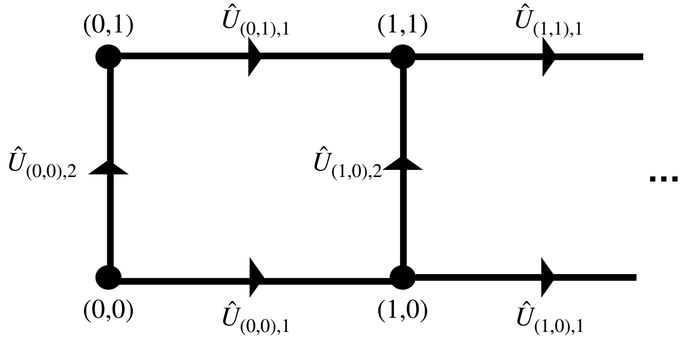
Gauge Invariance, Hilbert Space, Hamiltonian
Quantum Annealing

- Combine[1] ideas of using a **discrete group** G and the **representation basis, motivation:**
  - Training in new approaches
  - Perhaps learn something useful, since we can e.g. compare the approximation j_{max} with the full "story" of G
- Steps ?
  1. Define a system and "build" the Hilbert Space while preserving Gauge Invariance
  2. Compute the Hamiltonian
  3. Calculate the spectrum (ground-state)
  4. …

[1] MF, Philipsen, O. and Winterowd, Ch., arXiv: 2206.14679 [hep-lat]

Define a system …
Quantum Annealing

- Let's start with the choice of $G = D_n, n = 3,4\ldots$, i.e. the Dihedral group
  - Finite, $|G| = 2n$, isometries of the regular N-polygon
  - Non-Abelian, serves e.g. as truncation to $O(2)$ for $n \to \infty$ and n odd.
  - Already studied in the context of gate-based computing with excellent description[1]
  - Geometry: Let's start with a ladder of plaquettes …



$$H_{KS} = H_E + H_B$$

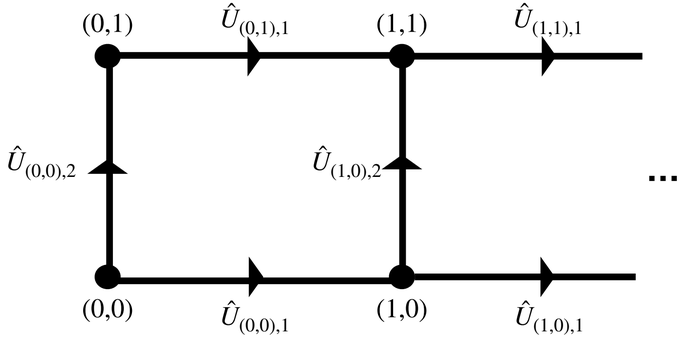
$$H_B = \lambda_B \sum_p \mathrm{Tr}(U_p + U_p^\dagger) \text{ and } H_E = \lambda_E \sum_x \sum_{i=1}^{d} \sum_{jmn} f_j |jmn\rangle_{x,i} |\langle jmn|_{x,i} .$$

[1] Lamm, H. et al., Phys. Rev. D 100, 034518 (2019)

…“build” gauge-invariant Hilbert Space …

Quantum Annealing

Start from the lattice with all links in the trivial rep $|000\rangle$



- Act[1] with plaquette $U_p^{(2)}$ using

$$\hat{U}^{(2)}_{m'n'}|jmn\rangle = \sum_J \sum_M \sum_N \sqrt{\frac{\dim(j)}{\dim(J)}} \times \langle 2m'jm\,|\,JM\rangle\langle JM\,|\,2n'jn\rangle\,|\,JMN\rangle$$

- Doing so in a systematic way creates an exact enumeration of $\mathscr{H}$ whose size grows exponentially with L.

- Filter those configs of $j's$ who additionally satisfy **Gauss’s law,** i.e. locally

$$|00\rangle_x = \sum_{m_I} \sum_{m_A} \sum_{m_E} (-)^{f(j_A,j_E,j_I;m_I)}\,|j_A m_A\rangle \otimes |j_E m_E\rangle \otimes |j_I m_I\rangle \begin{pmatrix} j_A & j_E & j_I \\ m_A & m_E & m_I \end{pmatrix}$$

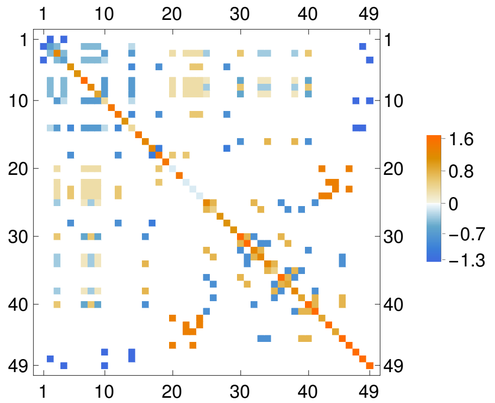
[1] Byrnes, Tim and Yamamoto, Yoshihisa, Phys. Rev. A73 (2006) 022328

Compute the Hamiltonian
Quantum Annealing

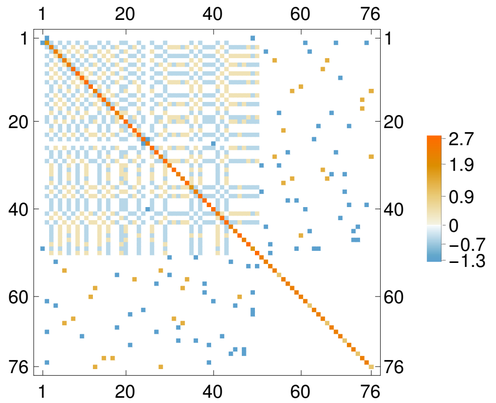
Once config space $\{j\}$ is labelled, compute $H_{ik} = \langle\psi_i|H_{KS}|\psi_k\rangle$ where $|\psi\rangle_{\{j\}} = \bigotimes_s |00\rangle_s(\{j\})$.

N	2	3	4
D_3	49	251	$O(1300)$
D_4	76	392	$O(2500)$

Table I. List of the size of the physical Hilbert space, N_{conf}, on a ladder of size N for D_3 and D_4. The configurations are enumerated by a set of integers $\{j_i, i = 1, 2, \ldots, 3N\}$ characterizing the irrep of each link on whereby Gauss's law is satisfied at each site.



D_3



D_4

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Spectrum ?
Quantum Annealing

- Ground-state calculation in variational formulation

$$E_0 \leq \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}\text{, where } |\psi\rangle = \sum_i a_i |\psi\rangle_i \text{ with } a_i \in \mathbb{R}$$

- Can be cast into QUBO problem using binary variables $q_i \in \{0,1\}$ suitable for Quantum Annealer[1,2]

$$a_\alpha^{(z+1)} = a_\alpha^{(z)} - \frac{q_{\alpha,K}}{2^z} + \sum_{i=1}^{K-1} \frac{q_{\alpha,i}}{2^{K-i+z}},$$

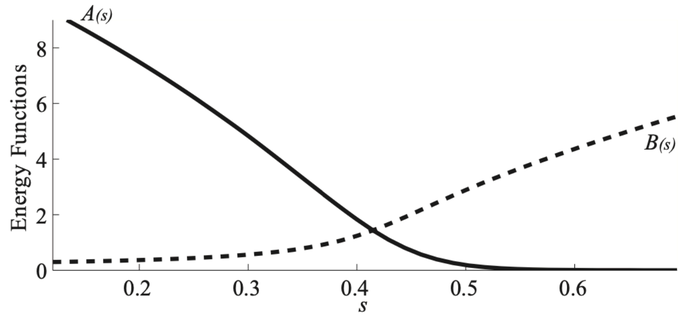
$$F = \langle\psi|\hat{H}|\psi\rangle - \eta\langle\psi|\psi\rangle = \sum_{\alpha,\beta}^{N_{\text{conf}}} \sum_{i,j}^{K} Q_{\alpha\beta,ij} q_{\alpha,i} q_{\beta,j}$$

[1] Rahman, S. et al., Phys. Rev. D 104, 034501 (2021)
[2] Illa, Marc and Savage, Martin J., arXiv: 2202.12340 [quant-ph]

Ground-State Calculation
Quantum Annealing

Computations done on D-Wave QA in forward annealing mode[1]

$$H_{QA} = A(s)\, H_I + B(s)\, H$$

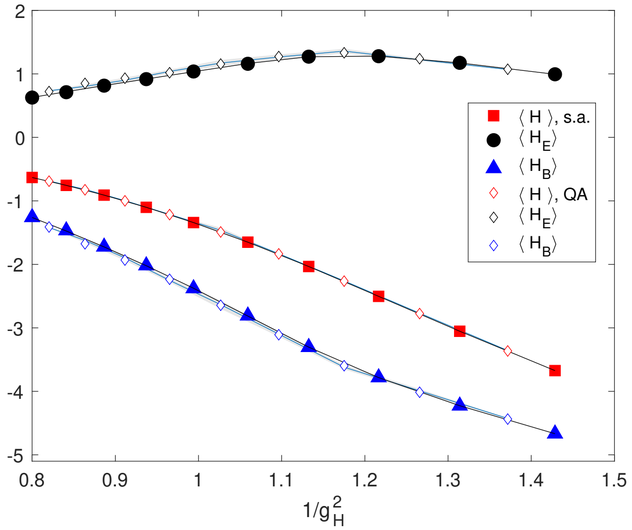


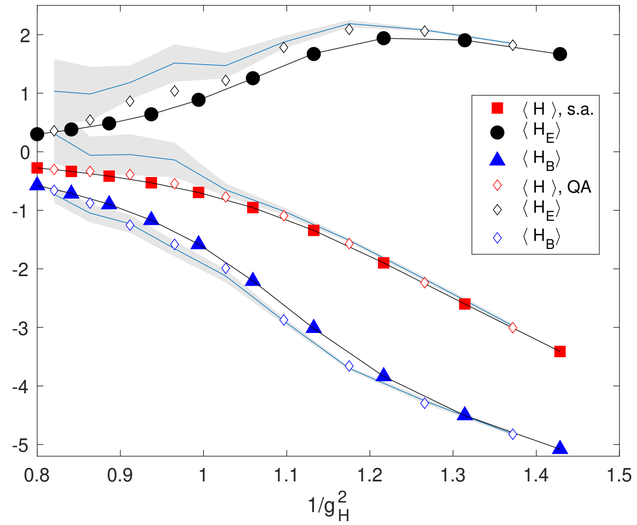
At end of annealing schedule extract ground-state coefficients a_i and hence $|\psi\rangle = \sum_i a_i |\psi\rangle_i$

Can extract

$$\langle O \rangle, O = H_B, H_E \ldots$$

Process does not always converge, results have hence uncertainty (grey band), visibly larger for the larger group D_4.





[1] Catherine C. McGeoch, Synthesis Lectures on Quantum Computing 2014 5:2, 1-93

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Thank you !